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Solitary waves and N coupled nonlinear equations

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Abstract. We present analytic sets of N solitary waves as solutions of N coupled nonlinear Schrödinger and of N coupled 'quadratic' equations, for which the N^2 nonlinear coupling parameters of the coupled equations can take up wide ranges of values. The coupled equations may not be integrable in the usual sense, but have analytic solitary waves and are solvable for the given initial conditions prescribed by the analytic solutions. Potential applications of these solitary waves are discussed.

For its many applications, especially in nonlinear optics [1], the problem of two coupled nonlinear Schrödinger (CNLS) equations has been studied for many years. Following the work of Manakov [2], coupled solitary waves that consist of bright–bright, bright–dark, and dark–dark pairs that can propagate in the normal or anomalous group-velocity dispersion (GVD) region have been found [3–6]. The simplest bright and dark solitary waves have the forms $\operatorname{sech}\xi$ and $\tanh\xi$ respectively, where $\xi = \alpha(t - z/v)$, α is some constant and t, z and v denote time, displacement and velocity. Solitary wave-pairs that consist of product types of the forms $\tanh(\alpha_1\xi)\operatorname{sech}^{s-1}(\alpha_2\xi)$ and $\operatorname{sech}^2(\alpha_1\xi)$, where $1 \leqslant s \leqslant 2$ have also been given [7–9]. Solitary waves each of which is a superposition of bright and dark solitary waves were given by the author [10], and many periodic solitary waves which are expressed in terms of Jacobian elliptic functions or their products were given by several authors [10–13]. Specific integrable parametric choices for two CNLS equations have been studied using Painlevé analysis [14].

In this paper, we consider N (>1) coupled nonlinear Schrödinger-like equations and present N complementary solitary waves, i.e. solitary waves of *different* waveform, given by $P^m_{N-1}(\tanh\xi)$, $m = 0, 1, \ldots, N-1$, where $P^m_n(x)$ is the associated Legendre function, as a solution of these N CNLS equations. We express the N^2 nonlinear coupling parameters in terms of $2N+1$ variable parameters for which these coupled equations have these solitary waves as a solution and are solvable for the initial conditions prescribed by the analytic solutions. We should note that the coupled equations are not, or may not be, integrable in the usual sense where integrable means for all possible initial conditions. One of the objectives of this paper is to bring our attention from integrable to solvable cases for certain given initial conditions because analytic solutions for the latter can be applied to many useful cases where the nonlinear coupling parameters can assume wide ranges of values while the integrable cases offer only very few choices of coupling parameters. For this considerable gain, we must use the more restricted initial conditions prescribed by these analytic solutions; but these conditions can often be experimentally simulated without too much problem. For $N = 2$, our solutions give the well known dark and bright solitary waves. We also present N complementary solitary waves $P^{2(k-1)}_{2(N-1)}(\tanh\xi)$, $k = 1, \ldots, N$, as a solution of N coupled quadratic (CQ) equations, which are an analogue of N CNLS equations with quadratic instead of cubic nonlinear couplings. Interesting implications and applications of these higher-order coupled solitary waves, which cannot be solitary individually, are discussed.

Consider a set of N coupled equations for the slowly varying complex envelopes or components $\phi_m(z,t)$, $m = 1, 2, \ldots, N$ of the electric fields propagating along the z-axis that satisfy the following coupled nonlinear Schrödinger-like equations

$$\mathrm{i}\phi_{mz} + \phi_{mtt} + \kappa_m\phi_m + \left(\sum_{n=1}^{N} p_{mn}|\phi_n|^2\right)\phi_m + \left(\sum_{n=1}^{N} q_{mn}\phi_n^2\right)\phi_m^* = 0 \qquad m = 1, \ldots, N, \tag{1}$$

where p, q and κ are parameters characteristic of the medium, and where the subscript m for different components of ϕ is to be distinguished from the subscripts in z and t which denote derivatives with respect to z and t, respectively. A closely related set of coupled equations is

$$\mathrm{i}\psi_{mz} + \psi_{mtt} + \left(\sum_{n=1}^{N} p_{mn}|\psi_n|^2\right)\psi_m + \left(\sum_{n=1}^{N} q_{mn}\psi_n^2\mathrm{e}^{2\mathrm{i}\kappa_n z}\right)\psi_m^*\mathrm{e}^{-2\mathrm{i}\kappa_m z} = 0 \qquad m = 1, \ldots, N \tag{2}$$

which can be transformed into (1) with the substitutions $\psi_m = \phi_m \exp(-\mathrm{i}\kappa_m z)$. We first search for the stationary-wave solution of the form

$$\phi_m(z,t) = x_m(t)\exp(\mathrm{i}\Omega z) \tag{3}$$

where Ω is a real constant, and $x_m(t)$ are real functions of t only. Equations (1) reduce to the following, which we call the associated dynamical coupled nonlinear Schrödinger equations:

$$\ddot{x}_m - A_m x_m + \left(\sum_{n=1}^{N} b_{mn}x_n^2\right)x_m = 0 \qquad m = 1, \ldots, N \tag{4}$$

where $\dot{x}$ denotes $\mathrm{d}x/\mathrm{d}t$, and where

$$A_m = \Omega - \kappa_{m'} \qquad \text{and} \qquad b_{mn} = p_{mn} + q_{mn}. \tag{5}$$

Since equations (1) and (2) are invariant under a Galilean transformation, the travelling waves can be constructed from (3) by replacing $\phi_m(z,t)$ by

$$\phi_m(z, t - z/v)\exp\{\mathrm{i}[t - z/(2v)]/(2v)\} \tag{6}$$

where v is the velocity of the waves.

An analogous set of coupled equations which we call N CQ equations is

$$\mathrm{i}\phi_{mz} + \phi_{mtt} + \kappa_m\phi_m + \left(\sum_{n=1}^{N} p_{mn}|\phi_n|\right)\phi_m + \left(\sum_{n=1}^{N} q_{mn}\phi_n\right)\phi_m^* = 0 \qquad m = 1, 2, \ldots, N. \tag{7}$$

The corresponding associated dynamical CQ equations are

$$\ddot{x}_m - A_m x_m + \left(\sum_{n=1}^{N} b_{mn}x_n\right)x_m = 0 \qquad m = 1, 2, \ldots, N \tag{8}$$

where we have used the same substitutions (3). To eliminate the permutation symmetry, we arrange equations (4) and (8) such that $A_1 \leqslant A_2 \leqslant \cdots \leqslant A_N$.

Let $P_n^m(x)$ denote the associated Legendre function of degree n and order m. We make the ansatz that

$$x_j = \sqrt{C_j}\,P_{N-1}^{j-1}[\tanh(\alpha t)] \qquad j = 1, 2, \ldots, N \tag{9}$$

Table 1. $P_n^m(\tanh\xi)$ for $n = 1$–4.

n	m	$P_n^m(\tanh\xi)$
1	0	$\tanh\xi$
	1	$\operatorname{sech}\xi$
2	0	$\operatorname{sech}^2\xi - \frac{2}{3}$
	1	$\tanh\xi\operatorname{sech}\xi$
	2	$\operatorname{sech}^2\xi$
3	0	$\tanh\xi(\operatorname{sech}^2\xi - \frac{2}{5})$
	1	$\operatorname{sech}\xi(\operatorname{sech}^2\xi - \frac{4}{5})$
	2	$\tanh\xi\operatorname{sech}^2\xi$
	3	$\operatorname{sech}^3\xi$
4	0	$\operatorname{sech}^4\xi - \frac{8}{7}\operatorname{sech}^2\xi + \frac{8}{35}$
	1	$\tanh\xi\operatorname{sech}\xi(\operatorname{sech}^2\xi - \frac{4}{7})$
	2	$\operatorname{sech}^2\xi(\operatorname{sech}^2\xi - \frac{6}{7})$
	3	$\tanh\xi\operatorname{sech}^3\xi$
	4	$\operatorname{sech}^4\xi$

where C_j is real and positive, for equations (4); and

$$x_j = C_j P_{2(N-1)}^{2(j-1)}[\tanh(\alpha t)] \qquad j = 1, 2, \ldots, N \tag{10}$$

where C_j is real, for equations (8). The sets of solitary waves $P_n^m(\tanh\xi)$ for $n = 1$–4, normalized such that the coefficient of the highest power of $\operatorname{sech}\xi$ is one, are given in table 1. To express our results in a compact form, we shall define three $N \times N$ matrices in the following. We first define $a_{ij}^{(N)}$ as follows.

For equation (4), $a_{ij}^{(N)}$ is the coefficient of $x^{2(i-1)}$ in $[P_{N-1}^{j-1}(x)]^2$ when it is expanded as

$$[P_{N-1}^{j-1}(x)]^2 = \sum_{i=1}^{N} a_{ij}^{(N)} x^{2(i-1)} \tag{11}$$

and for equation (8), $a_{ij}^{(N)}$ is the coefficient of $x^{2(i-1)}$ in $P_{2(N-1)}^{2(j-1)}(x)$ when it is expressed as

$$P_{2(N-1)}^{2(j-1)}(x) = \sum_{i=1}^{N} a_{ij}^{(N)} x^{2(i-1)}. \tag{12}$$

Our first $N \times N$ matrix is $\Gamma = [c_{ij}]$ whose matrix elements $c_{ij} = a_{ij}^{(N)} C_j$, where C_j is the coefficient in (9) for equation (4), and is the coefficient in (10) for equation (8). Our second matrix is $\mathbf{B} = [b_{ij}]$, where b_{ij} are the nonlinear coupling parameters given in equation (4) or (8). Our third matrix is $\mathbf{D} = [d_{ij}]$, where $d_{1j} = A_j + [(N-1)N - (j-1)^2]\alpha^2$, $d_{2j} = -(N-1)N\alpha^2$, $d_{3j} = d_{4j} = \cdots = d_{Nj} = 0$ for equation (4), and $d_{1j} = A_j + [(2N-2)(2N-1) - 4(j-1)^2]\alpha^2$, $d_{2j} = -(2N-2)(2N-1)\alpha^2$, $d_{3j} = d_{4j} = \cdots = d_{Nj} = 0$ for equation (8).

Substitutions of the ansatz (9) or (10) into equations (4) or (8) lead to N^2 algebraic equations which can be expressed conveniently in terms of the three matrices Γ, $\mathbf{B}$, and $\mathbf{D}$ as

$$\Gamma\mathbf{B}^T = \mathbf{D}$$

or

$$\mathbf{B}^T = \Gamma^{-1}\mathbf{D} \tag{13}$$

where $\mathbf{B}^T$ denotes the transposed matrix of $\mathbf{B}$. Provided that Γ^{-1} exists, equation (13) gives the set of parameters b_{ij} in equations (4) or (8) in terms of the A_j given in those equations, and in terms of the generally arbitrary amplitudes C_j in (9) or (10), i.e. equation (13) gives the N^2 nonlinear coupling parameters b_{ij} in terms of $2N + 1$ variable parameters A_j, C_j, $j = 1, \ldots, N$, and α, and for these b_{ij}, equations (4) or (8) are solvable.

If, on the other hand, we are given the set of N^2 values of b_{ij} and ask whether $2NA_j$ and C_j (C_j must be >0 for equation (4)) and α can be found that yield solutions (9) and (10), the answer would be no generally unless the given values of b_{ij} are such that $2N+1$ values of A_j, C_j and α can be found that satisfy the N^2 equations.

For the special case of $b_{ij} = \varepsilon$, for all $i, j = 1, \ldots, N$, where $\varepsilon = +1$ or -1, we can give the answer in a compact form. We write equation (13), in this case, as

$$\mathbf{a}\vec{C} = \vec{d} \tag{14}$$

where the $N \times N$ matrix $\mathbf{a} = [a_{ij}^{(N)}]$, the N-dimensional column vector $\vec{C} = \mathrm{col}(C_1, C_2, \ldots, C_N)$, and the N-dimensional column vector $\vec{d} = \mathrm{col}(d_{11}, d_{21}, \ldots, d_{N1})$. The consistency requirement becomes $N - 1$ equations on $A_2, \ldots, A_N$ which must be related to A_1 by

$$A_j = A_1 + (j-1)^2\alpha^2 \qquad j = 2, \ldots, N \tag{15a}$$

for equations (4); and

$$A_j = A_1 + 4(j-1)^2\alpha^2 \qquad j = 2, \ldots, N \tag{15b}$$

for equations (8).

Thus, if $\mathbf{a}^{-1}$ exists, and if the A in equations (4) or (8) are given by equation (15*a*) or (15*b*), then (9) and (10) are solutions of (4) and (8) respectively with C_j given by

$$\vec{C} = \mathbf{a}^{-1}\vec{d}. \tag{16}$$

For equations (4), there is a further restriction that the C_j given by equation (16) must all be positive.

Equations (9)–(16) complete the description of our solutions for equations (4) and (8), and with the use of transformation (6), our N complementary solitary-wave solutions for N CNLS and CQ equations (1) and (7). It will be noted that the N complementary solitary waves for CNLS equations consist of symmetric (about $\xi = 0$) as well as antisymmetric waves, and those for CQ equations consist of only symmetric waves.

We shall illustrate our results first with the important example of $N = 2$ for equation (1). Using equation (13), we find that equations (4) are solvable (with solutions given by equation (9)) if the b are given in terms of A_1, A_2, C_1, C_2, and α by

$$\begin{aligned} &b_{11} = A_1C_1^{-1} \qquad b_{12} = (A_1 + 2\alpha^2)C_2^{-1} \\ &b_{21} = (A_2 - \alpha^2)C_1^{-1} \qquad b_{22} = (A_2 + \alpha^2)C_2^{-1}. \end{aligned} \tag{17}$$

As the parameters A, C and α can be considered as variables, equations (17) give a wide range of values that the nonlinear coupling parameters b can have for which the two CNLS equations are solvable, and have a coupled solitary-wave solution given by equation (9).

A simpler result follows if we choose $A_1 = A_2 = -C_2$, $\alpha^2 = C_2 - C_1$, for which we find

$$b_{11} = -C_2C_1^{-1} \qquad b_{12} = 1 - 2C_1C_2^{-1} \qquad b_{21} = 1 - 2C_2C_1^{-1} \qquad b_{22} = -C_1C_2^{-1}. \tag{18}$$

The two coupled NLS equations characterized by (18) can be shown to be mathematically equivalent to the coupled equations that arise from the Maxwell–Schrödinger equations that describe the resonant interactions of two electromagnetic waves with a three-level system that was studied by Hioe and Grobe [15].

For the case of $b_{ij} = \varepsilon = +1$ or -1, for all $i, j = 1, \ldots, N$, solution (9) of equations (4) gives, for $N = 2$,

$$\begin{aligned} &x_1 = \sqrt{C_1}\tanh\alpha t \qquad x_2 = \sqrt{C_2}\,\mathrm{sech}\,\alpha t \\ &C_1 = \varepsilon A_1 \qquad C_2 = \varepsilon(2A_2 - A_1) \qquad \alpha^2 = A_2 - A_1 \\ &A_2 > A_1 > 0(\varepsilon = +1) \qquad A_2 > A_1 < 0(\varepsilon = -1) \end{aligned} \tag{19}$$

and for $N = 3$,

$$\begin{aligned} &x_1 = \sqrt{C_1}(\mathrm{sech}^2\alpha t - \tfrac{2}{3}) \qquad x_2 = \sqrt{C_2}\tanh\alpha t\,\mathrm{sech}\,\alpha t \qquad x_3 = \sqrt{C_3}\,\mathrm{sech}^2\alpha t \\ &C_1 = 9\varepsilon A_1/4 \qquad C_2 = 3\varepsilon(2A_2 - A_1) \qquad C_3 = 3\varepsilon(8A_2 - 7A_1)/4 \\ &\alpha^2 = A_2 - A_1 \qquad A_3 = 4A_2 - 3A_1 \\ &A_2 > A_1 > 0(\varepsilon = +1) \qquad A_2 > A_1 \geqslant 8A_2/7 < 0(\varepsilon = -1). \end{aligned} \tag{20}$$

Next we illustrate our results with the example of $N = 2$ for equation (7). Equations (8) are solvable with solutions given by equation (10) if the b's are given in terms of the A_1, A_2, C_1, C_2, and α by

$$\begin{aligned} &b_{11} = -\tfrac{3}{2}A_1C_1^{-1} \qquad b_{12} = \tfrac{3}{2}(A_1 + 4\alpha^2)C_2^{-1} \\ &b_{21} = -\tfrac{3}{2}(A_2 - 4\alpha^2)C_1^{-1} \qquad b_{22} = \tfrac{3}{2}A_2C_2^{-1}. \end{aligned} \tag{21}$$

Equations (21) give a wide range of values that the nonlinear coupling parameters b's can have for which the two CQ equations are solvable, and have a coupled solitary-wave solution given by equation (10).

For the case of $b_{ij} = \varepsilon$ for all $i, j = 1, \ldots, N$, solution (10) of equation (8) gives, for $N = 2$,

$$\begin{aligned} &x_1 = C_1(\mathrm{sech}^2\alpha t - \tfrac{2}{3}) \qquad x_2 = C_2\,\mathrm{sech}^2\alpha t \\ &C_1 = -3\varepsilon A_1/2 \qquad C_2 = 3\varepsilon A_2/2 \qquad \alpha^2 = (A_2 - A_1)/4 \end{aligned} \tag{22}$$

and for $N = 3$,

$$\begin{aligned} &x_1 = C_1(\mathrm{sech}^4\alpha t - \tfrac{8}{7}\mathrm{sech}^2\alpha t + \tfrac{8}{35}) \qquad x_2 = C_2\,\mathrm{sech}^2\alpha t(\mathrm{sech}^2\alpha t - \tfrac{6}{7}) \\ &x_3 = C_3\,\mathrm{sech}^4\alpha t \\ &C_1 = 35\varepsilon A_1/8 \qquad C_2 = -35\varepsilon A_2/6 \qquad C_3 = 35\varepsilon A_3/24 \\ &\alpha^2 = (A_2 - A_1)/4 \qquad A_3 = 4A_2 - 3A_1 \\ &A_2 > A_1 > 0 \qquad (\text{for } \varepsilon = +1) \qquad A_2 > A_1 < 0 \qquad (\text{for } \varepsilon = -1). \end{aligned} \tag{23}$$

Let us consider further the CNLS equations for the case of $b_{ij} = \varepsilon$ for all $i, j = 1, \ldots, N$. The case $\varepsilon = +1$, $N = 1$ can be identified with the standard NLS equation that gives the bright solitary wave, and the case $\varepsilon = -1$, $N = 1$ can be seen to be equivalent to the standard equation that gives the dark solitary wave. For $N > 1$, our solutions give sets of complementary solitary waves, i.e. solitary waves of *different* waveform. Let us refer to the N complementary waves for the N coupled equations as waves of order N. Other solutions can be constructed in two following ways.

(I) For a given set of N coupled equations (4), depending on $\varepsilon = +1$ or -1, various solutions can be constructed which consist of two or more waves of the *same* form of order N or lower, i.e. the N solitary waves are no longer entirely complementary but may consist of two or more identical waveforms. It should be noted that (a) not every combination is a possible solution, e.g. the bright–bright solitary wave-pair is not a solution of CNLS equations for $N = 2$, $\varepsilon = -1$ and thus the pair cannot propagate in the normal GVD region; and (b) when two or more of the N waves are of the same form, it necessarily requires the corresponding values of the A's in equations (4) to be equal.

(II) Besides waves of order N for N coupled equations, another set of N complementary-wave solutions can be obtained as follows. For $\varepsilon = +1$, we may set $C_1 = 0$ in the set of solution for $N+1$ coupled equations, i.e. we use the set of waves of order $N+1$ excluding the first one, and find that they are a possible solution of N coupled equations for $\phi_2, \ldots, \phi_{N+1}$ (which we may relabel $\phi_1, \ldots, \phi_N$). For $\varepsilon = -1$, we may set $C_{N+1} = 0$, and find the remaining N complementary waves of order $N+1$ to be a solution of N coupled equations for $\phi_1, \ldots, \phi_N$. However, a wave of order $N+1$ cannot be a solution of coupled equations involving $N-1$ or less coupled field envelopes or components. In particular, any wave or order >2 is not by itself a solitary wave of an NLS equation.

A potentially interesting implication of (II) above can be illustrated with the following specific examples for CNLS equations. Let us denote the three complementary waves of order 3 (see table 1 and equation (20)) by $g_1 = \operatorname{sech}^2\xi - \frac{2}{3}$, $g_2 = \tanh\xi \operatorname{sech}\xi$, $g_3 = \operatorname{sech}^2\xi$. It can be checked that the pair of solitary waves (g_2, g_3) can propagate in the anomalous GVD region ($\varepsilon = +1$) but not in the normal GVD region ($\varepsilon = -1$), and that the pair of solitary waves (g_1, g_2) can propagate in the normal GVD region but not in the anomalous GVD region. However, by having either pair of waves coupled to a third complementary wave, the three coupled waves (g_1, g_2, g_3) can propagate in the normal or the anomalous GVD region, by having appropriate values of A in equations (4). We have a similar situation when we go from $N = 1$ to $N = 2$: the bright solitary wave is a solution for $N = 1$, $\varepsilon = +1$ and not $\varepsilon = -1$, while the dark solitary wave is a solution for $N = 1$, $\varepsilon = -1$ and not $\varepsilon = +1$, but the coupled bright and dark solitary wave-pair can propagate in either the normal or anomalous GVD region for both waves, i.e. the bright–dark solitary wave-pair is a solution for $N = 2$ for $\varepsilon = +1$ or -1. Similar examples can be shown for the CQ equations. We have thus extended the very successful idea of using two optical waves instead of one for better control of wave propagation [15–17] to that of using $N+1$ optical waves instead of N.

In summary, we have presented new solitary wave-sets that give N complementary solitary waves for N CNLS and CQ equations (1) and (7). The N^2 nonlinear coupling parameters b_{ij} are expressed in terms of $2N+1$ variable parameters $A_1, \ldots, A_N$, $C_1, \ldots, C_N$, and α by equation (13) for which the coupled equations (4) and (8) have N coupled solitary-wave solutions (9) and (10) respectively, and for which the coupled nonlinear equations (1) and (7) are thus solvable. The novel feature of these results is that not only new solitary waveforms have been found, but also the introduction of (i) new generations of coupled solitary waves which cannot be solitary individually, and the idea that (ii) increasing the number of coupled waves may indeed extend the region of validity for propagation of solitary waves, and that (iii) there can be a wide and continuous range of nonlinear coupling parameters for which coupled nonlinear equations have analytic solitary waves. A recent experimental observation of multihump solitons [18] may encourage experimental creation and observation of the new solitary waves given here.

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